F I G
XX INTERNATIONAL CONGRESS
MELBOURNE, AUSTRALIA, 1994

MINIMUM ERROR MAP PROJECTIONS SUITABLE FOR G.I.S. IN AUSTRALIA.

PROJECTION APPROPRIÉE DES CARTES MINIMUM ERROR ENGLOBANT L'AUSTRALIE POUR G.I.S.

KARTENPROJEKTIONEN MIT DEN GERINGSTEN FEHLERN GEEIGNET FÜR G.I.S. IN AUSTRALIEN.

R.E. Deakin Department of Land Information, Royal Melbourne Institute of Technology, GPO Box 2476V, Melbourne, 3001, AUSTRALIA.

SUMMARY

The Universal Transverse Mercator (UTM) projection is the basic projection used for mapping and survey coordination in Australia. Due to restrictions imposed by zone width, UTM coordinates may not be suitable for simple manipulations in Geographic Information System (GIS) databases covering regions having large east-west extent. This problem can be overcome by using coordinates derived from appropriate *minimum-error* map projections covering the region in one complete zone. Such projections minimise the usual areal, angular and linear distortions associated with any projection of the Earth.

RÉSUMÉ

La projection Universal Transverse Mercator (UTM) est la projection de base utiliseé eu cartographie et pour le levé de plans en Australie. A cause des restrictions dues á la largeur de la zone, la manipulation simple des coordonnés UTM peut s'avérer mal adaptée aux indications typographiques GIS de régions ayant une grande superficie de l'est á l'ouest, le probléme peut être surmonté en employant des donneés provenant de la projection appropricé de cartes Minimum Error englobant l'Australie en une seule zone. Je telles projections réduisent au minimum les distorsions territoriales, angulaires et linéaires qui sont associées á toutes projections de la Terre.

ZUSAMMENFASSUNG

Die universale Merkator Querprojektion (UTM) is die fundamentale Projektion benützt für Kartierung und Vermessungskoordination in Australien. Wegen Beschränkungen von der Zonenbreite, UTM Koordinaten sind nicht immer geeignet für einfache Manipulation in G.I.S. Datengrundlagen in Gebieten mit grossen Ost-West Umfang. Dieses Problem kann ü berkommen werden mit der Benüzung von Koordinaten abgeleitet von Minimalfehler Karten Projektionen, die Australien in einer Zone decken. Solche Projektionen verringern die Fehler der gebräuchliche Flächen, Winkel und Linearverzerrungen verbunden mit jeder Projektion auf der Erdoberfläche.

1. INTRODUCTION

In Australia, all mapping and survey coordination is based on a Transverse Mercator projection of latitudes and longitudes on the Australian Geodetic Datum and for national and state mapping purposes, the Australian Map Grid (AMG) has been defined to correspond with the Universal Transverse Mercator Grid. Each AMG zone is 6° wide with a central scale factor of 0.9996 and a coordinate origin 10,000,000 metres south and 500,000 metres west of the intersection of the equator and the central meridian of the zone. Australia is covered by eight such zones and with the exception of the Australian Capital Territory, no State or Territory is completely covered by a single AMG zone. In addition to the AMG, the state of New South Wales has adopted (for the purposes of survey coordination) the Integrated Survey Grid (ISG) with 2° zones, a central scale factor of 0.99994 and a coordinate origin 5,000,000 metres south and 300,000 metres west of the intersection of the equator and the central meridian of the zone. Each AMG zone contains three ISG zones, the second of which shares the same central meridian as the AMG zone, and seven ISG zones cover New South Wales. The AMG and the ISG have been used by all States and Territories of the Commonwealth of Australia for the exchange of coordinated mapping data and as a consequence; Federal, State and Local government organisations, government business enterprises, and private business companies have extensive data bases linked to AMG and ISG coordinates.

Geographic Information Systems (GIS) offer sophisticated data analysis techniques and are increasingly used by Local, State and Federal authorities, as well as private organisations, to assist in management and planning. If a GIS is used to analyse and display data in a region covered by a single AMG zone then the data can be linked to AMG coordinates for that zone, but if the region under analysis lays within two or more zones, data will have to be coordinated in a different system since AMG and ISG coordinates are related to a map projection which is unsuitable for display and analysis of large areas as a single entity. Some GIS offer a selection of map projections and coordinate transformation routines for this purpose.

Choosing a suitable map projection for a particular GIS application requires a knowledge of the inherent errors in projecting the curved surface of the Earth (sphere or ellipsoid) onto a plane, where such errors manifest themselves as areal, angular and length distortions. These common distortion measures, together with the developable surfaces upon which the Earth is projected, have been used to classify maps into groups with desirable properties such as *equal-area, conformal* and *equidistant* projections of the *cylindrical, conical* and *azimuthal* classes. Other projections, not fitting into these general classes, could be classified as *pseudocylindrical, pseudoazimuthal, pseudoconical* or *conventional* and often possess some desirable distortion characteristic or pleasing general appearance. Another group of map projections can be classified as *minimum-error*. Minimum-error projections are determined by selecting mathematical functions of projection parameters which measure the "error" in a map projection and those parameters varied to produce a map for which the selected error function is a minimum. Minimum-error projections can also be derived to have additional properties such as equal-area, conformal or equidistant.

This paper shows the development of a *minimum-error equal-area* map projection for the State of Victoria which is a significant improvement over two commonly used projections of the State. The technique used in this paper was proposed by Peters (1984) in his development of distance related world maps and also used by Canters (1989 and 1991).

2. MINIMUM-ERROR MAP PROJECTIONS

Sir George Airy (1861) proposed a method of determining projection parameters such that the sum of the squares of the scale errors, in the principal directions, summed for every point on the map is a minimum. Airy called his method "Balance of Errors" and applied it to an azimuthal projection. Col. Sir Henry James and Capt. A.R. Clarke (1862) corrected an error in Airy's calculations and applied the method to a general perspective projection of the Earth and A.E. Young (1920), extended the method to the general conical projection and demonstrated how the technique could be used to obtain the minimum-error projection of a particular class of projections. The minimum-error function, proposed by Airy and used by Young can be derived in the following manner:

> Tissot showed that an infinitesimal unit circle on the Earth will be projected as an ellipse on the map projection and that the lengths of the semi-axes of this *Indicatrix* are the scales in the principal directions of the map projection. If a and b are the lengths of the semi-axes of Tissot's Indicatrix, then $(1 - a)$ and $(1 - b)$ are scale errors, and since a and b are functions of the latitude (ϕ) and longitude (λ), the sum of the squares of the scale errors at a point is given by the function

$$f(\phi,\lambda) = \left[(1-a)^2 + (1-b)^2\right] \quad (1)$$

Summing the scale errors over the surface of the sphere represented on the map, leads to the integral

$$\iint f(\phi,\lambda)\,da$$

where $da = R^2\cos\phi\,d\phi\,d\lambda$ is the elemental area on the sphere of radius R. For R equal to unity *Airy's* minimum-error function can be expressed as

$$M = \int_{\lambda_1}^{\lambda_2}\int_{\phi_1}^{\phi_2}\left[(1-a)^2 + (1-b)^2\right]\cos\phi\,d\phi\,d\lambda \qquad (2)$$

Equations for minimum-error projections may be determined by expressing a and b as functions of the projection parameters and solving for those parameters by minimising M. Young (1920, pp.33-35) shows a method for obtaining minimum-error Mercator (conformal) and minimum-error Plate-Carree (equidistant) projections.

A method of determining minimum-error functions for *conformal* projections, uses a theorem developed by Gauss who showed that the necessary and sufficient condition for a conformal transformation from the spheroid (or sphere) to the plane is given by the complex expression (Lauf, 1983)

$$y + ix = f(\chi + i\lambda) \qquad (3)$$

where the function $f(\chi + i\lambda)$ is *analytic* and contains the *isometric* parameters χ (isometric latitude) and λ (longitude), and $i^2 = -1$. A necessary condition for an analytic function is that the *Cauchy-Riemann* equations are satisfied, ie.

$$\frac{\partial y}{\partial \chi} = \frac{\partial x}{\partial \lambda} \quad \text{and} \quad \frac{\partial y}{\partial \lambda} = -\frac{\partial x}{\partial \chi} \qquad (4)$$

Using this theorem, any conformal projection (x,y) (isometric parameters), can be transformed into another conformal projection (E,N) (also isometric parameters) by the complex expression

$$N + iE = f(y + ix) \qquad (5)$$

A function, $f(y+ix)$ which satisfies the Cauchy-Riemann equations is the complex polynomial

$$N + iE = \sum_{p=1}^{n}(A_p + iB_p)(y + ix)^p \qquad (6)$$

If k is the scale factor on the initial conformal projection (x,y), Snyder (1984, p.34) shows that the scale factor K on the transformed conformal map (E,N) is

$$K = \left|\sum_{p=1}^{n} p(A_p + iB_p)(y + ix)^{p-1}\right| k \qquad (7)$$

If $(K-1)$ is the scale error, a function E can be formed which is the sum of the squares of the scale errors for m points on the transformed map

$$E=\sum_{r=1}^{m}(K_r-1)^2 = \text{ minimum} \qquad (8)$$

Since E is a function of the unknown coefficients A and B for the m points, the *least-squares* technique can be used to determine the coefficients such that E is a minimum for the selected points. This technique has been used by Snyder (1984) to produce a low-error conformal map of the United States and also by Reilly (1973) who developed a minimum-error conformal projection for New Zealand. A feature of these minimum-error conformal projections is the ability to determine the coefficients from points so selected as to produce lines of constant scale factor, or *isocols,* which follow the shapes of countries or regions of interest.

Another method of deriving minimum-error projections (the one used in this paper) is based on a comparative distance function which measures the "error" in a distance between two points computed from map coordinates as compared to the distance on the Earth (sphere or spheroid) between the corresponding points. This function, proposed by Peters (1984) and used by Canters (1989 and 1991), is the *linear distortion* D, where

$$D_{ik}=\frac{|S_{ik}-s_{ik}|}{|S_{ik}+s_{ik}|} \qquad (9)$$

and

$$s_{ik}=\cos^{-1}\{\sin\phi_i\sin\phi_k+\cos\phi_i\cos\phi_i\cos(\lambda_k-\lambda_i)\} \qquad (10)$$

is the distance between two points P_i and P_k on the spherical Earth, and

$$S_{ik}=\left\{(x_k-x_i)^2+(y_k-y_i)^2\right\}^{\frac{1}{2}} \qquad (11)$$

is the distance between the two corresponding points on the map projection.

Canters (1989) shows that the linear distortion D can be converted to a corresponding scale factor K as follows

$$\frac{|1-K|}{|1+K|}=D \qquad (12)$$

which leads to a quadratic equation in K which can be solved for two reciprocal solutions

$$K_1=\frac{|1+D|}{|1-D|} \quad \text{and} \quad K_2=\frac{|1-D|}{|1+D|} \qquad (13)$$

with $\quad 0<K_2\le 1\le K_1$

If distances S and s are computed between n pairs of points, the *mean linear distortion* is

$$\overline{D} = \frac{\sum_{r=1}^{n} D_r}{n} \tag{14}$$

Canters (1989) expresses the rectangular projection coordinates (x,y) as polynomial functions of the geographical coordinates (ϕ,λ) in the form

$$x = \sum_{i=0}^{p}\sum_{k=0}^{p} C_{ik}\,\phi^i\,\lambda^k \tag{15}$$

$$y = \sum_{i=0}^{p}\sum_{k=0}^{p} C'_{ik}\,\phi^i\,\lambda^k \tag{16}$$

which gives the mean linear distortion for a given set of points as a function of the polynomial coefficients, $\overline{D} = f(C,C')$. This function can be minimised by a suitable choice of polynomial coefficients C and C'.

The three techniques outlined above, that are employed to form *minimum-error functions* (which can then be used to compute parameters of minimum-error map projections) are only a selection of the methods used and documented in cartographic literature. An excellent treatment of minimum-error map projections can be found in Snyder (1985), who details other methods and variations, and Dyer and Snyder (1989) develop a minimum-error equal-area map projection of Alaska along similar lines to the one presented here.

3. EQUAL-AREA MINIMUM ERROR PROJECTIONS

For various reasons, it may be desirable to produce a minimum-error projection with a particular distortion characteristic such as equal-area, conformal or equidistant. For a *minimum-error equal-area* projection, the minimum-error function must satisfy an "equal-area condition" and such a condition can be determined by considering the differential relationships known as the *Gaussian Fundamental Quantities, e, f* and g and the related quantity j as follows:

For a transformation from the sphere or spheroid (ϕ,λ) to the projection plane (x,y) such that $x = f_1(\phi,\lambda)$ and $y = f_2(\phi,\lambda)$, Lauf (1983), gives e, f, g and j for the plane as

$$e = \left(\frac{\partial x}{\partial \phi}\right)^2 + \left(\frac{\partial y}{\partial \phi}\right)^2 \tag{17}$$

$$f = \frac{\partial x}{\partial \phi}\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \phi}\frac{\partial y}{\partial \lambda} \tag{18}$$

$$g = \left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2 \qquad (19)$$

$$j = \sqrt{eg - f^2} = \left|\frac{\partial x}{\partial \phi}\frac{\partial y}{\partial \lambda} - \frac{\partial y}{\partial \phi}\frac{\partial x}{\partial \lambda}\right| \qquad (20)$$

with an element of area on the projection plane given by

$$da = j\, d\phi\, d\lambda \qquad (21)$$

Using similar differential relationships; if a transformation is made <u>in the plane</u> between the (*x*,*y*) system and the (*X*,*Y*) system such that $X = f_3(x, y)$ and $Y = f_4(x, y)$, where an element of area in the (*x*,*y*) plane is $da = dx\, dy$; then the corresponding element of area in the (*X*,*Y*) plane is $dA = j\, dx\, dy$ where $j = \left|\partial X/\partial x\ \partial Y/\partial y - \partial X/\partial y\ \partial Y/\partial x\right|$.

For an equal-area transformation then *dA* must equal *da*, which leads to the <u>equal-area condition in the plane</u>

$$\frac{\partial X}{\partial x}\frac{\partial Y}{\partial y} - \frac{\partial Y}{\partial x}\frac{\partial X}{\partial y} = 1 \qquad (22)$$

There are many equal-area transformations in the plane which satisfy equation (22). One such set of transformations may be derived by as follows:

let $X = f(x)$, ie, *X* is a function of *x* only, then $\dfrac{\partial X}{\partial x} = f'(x)$ and $\dfrac{\partial X}{\partial y} = 0$.

Equation (22) becomes $f'(x)\dfrac{\partial Y}{\partial y} = 1$ and solving for *Y* by integration gives

$$Y = \int \frac{dy}{f'(x)} = \frac{y}{f'(x)}$$

Similar reasoning can be used to derive an expression for *X* when $Y = g(y)$. These equal-area transformations in the plane can be summarised as

$$X = f(x), \qquad Y = \frac{y}{f'(x)} \qquad (23)$$

$$X = \frac{x}{g'(y)}, \qquad Y = g(y) \qquad (24)$$

Now, consider an equal-area projection of the sphere (ϕ,λ) to the plane (x,y), such as *Alber's* equal-area conic, where x and y are the "base projection coordinates" and

$$x = f_1(\phi,\lambda) \quad \text{and} \quad y = f_2(\phi,\lambda)$$

Using a 4th order polynomial and equations (23), an equal area transformation in the plane from (x,y) to (X,Y) is given by

$$X = f(x) = A_1 x + A_2 x^2 + A_3 x^3 + A_4 x^4 \tag{25}$$

$$Y = \frac{y}{f'(x)} = \frac{y}{A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3} \tag{26}$$

This projection is symmetric about the *X-axis* and preserves the scaling of the parallels along the central meridian. A second transformation in the plane between (X,Y) and (X',Y') using another 4th order polynomial and equation (24) is given by

$$Y' = B_1 Y + B_2 Y^2 + B_3 Y^3 + B_4 Y^4 \tag{27}$$

$$X' = \frac{X}{f'(Y')} = \frac{X}{B_1 + 2B_2 Y + 3B_3 Y^2 + 4B_4 Y^3} \tag{28}$$

This second projection in the plane is symmetric about the *Y'-axis*.

Using coordinates from equations (27) and (28) for n "control points" in a region, the mean linear distortion $\overline{D}$ is a function of the polynomial coefficients A_k and B_k. By using a function optimisation technique known as the *downhill simplex method* (Nelder and Mead, 1965) coefficients can be determined such that the function $\overline{D}$ is minimised.

4. RESULTS FOR VICTORIA

To develop a *Minimum-error equal-area* projection of Victoria, the base projection was chosen as *Alber's* equal-area projection of the sphere (Snyder, 1987, pp.98-103) with standard parallels $\phi_1 = -38°$ and $\phi_2 = -36°$. The radius of the sphere was taken as the mean radius of curvature for the Australian National Spheroid at latitude $\phi = -37°$ giving $R_m = 6372225m$ (nearest metre). Forty-one control points were chosen across Victoria at one-degree intervals of latitude and longitude and also where meridians crossed state borders. These $n = 41$ points give rise to $n(n-1)/2 = 820$ possible distances.

The mean linear distortion for *Alber's* projection over the 820 distances was

$$\overline{D}_{ALBERS} = 0.000037487 \quad \text{with scale factors: } K_1 = 1.000075 \text{ and } K_2 = 0.999925$$

Using equations (25) to (28) the function $\overline{D}$ was evaluated for the *Minimum-error equal-area* projection (X',Y') and minimised by solving for the coefficients A_k and B_k using the *simplex method* of Nelder and Mead given by Press, et al., (1992, pp.408-412) to give

$$\overline{D}_{MINERROR} = 0.000029575 \quad \text{with scale factors: } K_1 = 1.000059 \text{ and } K_2 = 0.999941$$

$$A_k = \begin{bmatrix} 1.039832888 \\ -0.000055433 \\ -0.019731969 \\ 0.187676360 \end{bmatrix} \qquad B_k = \begin{bmatrix} 1.039709031 \\ -0.000102416 \\ 0.059322526 \\ 0.178162285 \end{bmatrix}$$

Whilst the mean linear distortion values for *Alber's* and the *Minimum-error* projection are very small numbers, the percentage change in $\overline{D} = 78\%$. The values for K_1 and K_2 represent the mean scale factors for the 820 distances and signify that, on average, projection distances are enlarged (or reduced) by 75 parts per million (ppm) for *Alber's* projection and 59 ppm for the *Minimum-error equal-area* projection.

The maximum and minimum scale factors, (a and b respectively) for a map projection are the semi-axes of Tissot's *Indicatrix*. a and b can be computed from the *Gaussian fundamental quantities* e, f and g in the following manner given by Lauf (1983, p.74)

$$a^2 = \frac{(e'+g'+w)}{2} \quad \text{and} \quad b^2 = \frac{(e'+g'-w)}{2} \tag{29}$$

where

$$e' = \frac{e_{MAP}}{e_{SPHERE}}, \quad f' = \frac{f_{MAP}}{f_{SPHERE}} \quad \text{and} \quad g' = \frac{g_{MAP}}{g_{SPHERE}} \tag{30}$$

and

$$w^2 = (e'-g')^2 + 4f'^2 \tag{31}$$

e, f and g for the sphere are R^2, $R^2\cos\phi$ and $R^2\cos^2\phi$ respectively and e, f and g for the map projection are calculated from differential relationships similar to equations (17), (18) and (19) where partial derivatives are calculated numerically by computing the small changes in projection coordinates (dX',dY') caused by small changes in ϕ and λ ($d\phi = d\lambda = 0.001°$).

For *Alber's* and the *Minimum-error equal-area* projection, *a* and *b* were computed at 0.1° intervals of ϕ and λ covering Victoria and *isocols,* or "contours" of equal maximum scale factor, interpolated. Figures 1 and 2 show isocols (in intervals of 100 ppm) on *Alber's* projection and the *Minimum-error equal-area* projection respectively and hatching indicates regions where the maximum scale factor is less than 100 ppm. In Fig.1, the regions cover approximately 38% of the State, whilst in Fig.2, the hatched region is approximately 58% of the State.

These results indicate that the *Minimum-error equal-area* projection developed above, is a significant improvement over *Alber's* projection in the following ways:

(i) the percentage of the State's area bounded by a maximum scale error of 100 ppm is increased and combined in one region, and

(ii) on average, plane distances computed from projection coordinates will be "closer" to the corresponding true distance on the Earth.

In this study, several other equal-area polynomial transformations were investigated. Using *Alber's* projection as the base coordinates, 6th and 8th order polynomial transformations showed no significant improvement in the mean linear distortion $\overline{D}$, or in the pattern of isocols, over the 4th order polynomial transformation given above. Using a *Cylindrical equal-area* projection as base coordinates, a 4th order polynomial transformation gave a value of $\overline{D}$ significantly worse than that for *Alber's* projection and an undesirable pattern of isocols. This indicated that the choice of base projection coordinates was an important factor in developing the best minimum-error equal-area projection for a region.

5. CONCLUSION

This paper shows that simple polynomial transformations of *Alber's* projection coordinates can lead to a *minimum-error equal-area* projection with improved scale characteristics. Both *Alber's* and the *Minimum-error* projection are capable of mapping the State of Victoria in a complete zone and have decided advantages over the UTM projection with 6° zones (the base projection for the AMG) and other Transverse Mercator projections of the State. As a comparison, Figure 3 shows a Transverse Mercator projection of Victoria with a zone width of 12°. Lines of scale factor (ppm), indicate that distances computed on this projection would be badly distorted in many areas of the State.

The study also revealed that the selection of the base projection coordinates is an important factor in determining the "best" minimum-error projection for a region. It may be possible to derive a better projection if the standard parallels of *Alber's* projection were modified or if another projection (eg. *azimuthal equal-area*) was chosen as the base projection.

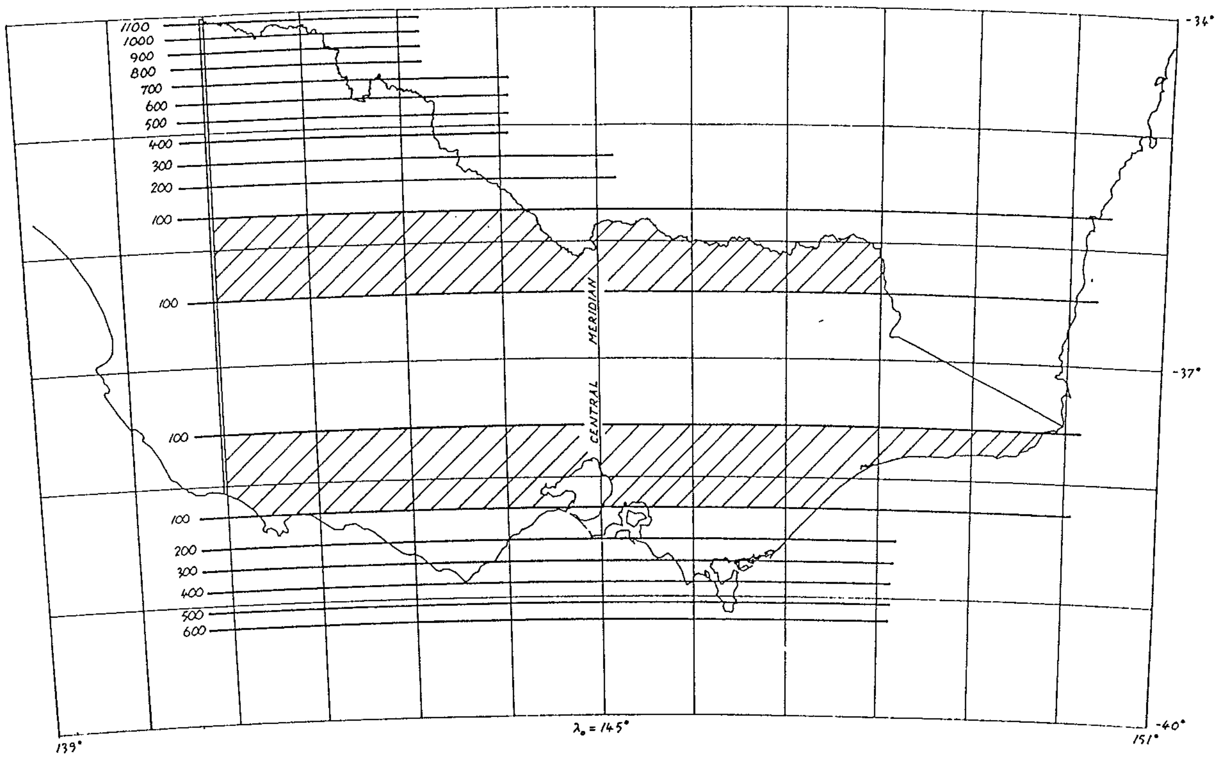

Figure 1. Alber's equal-area projection

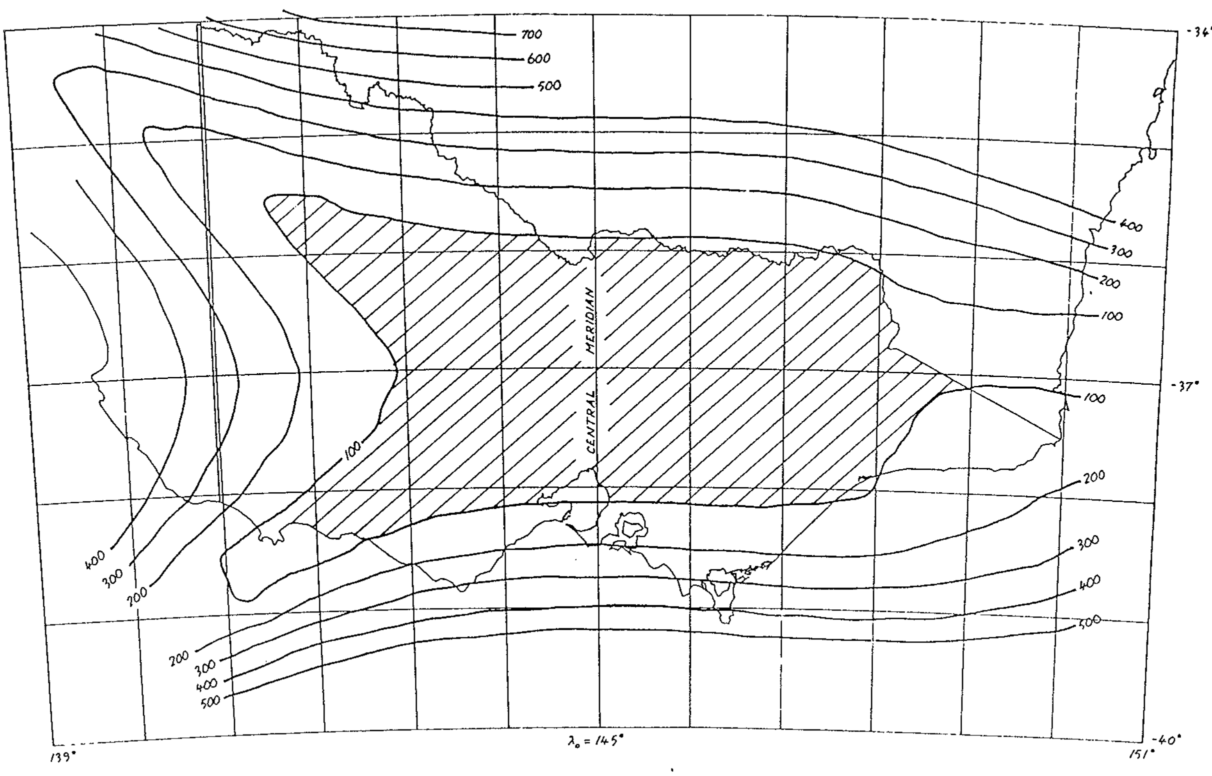

Figure 2. Minimum-error equal-area projection

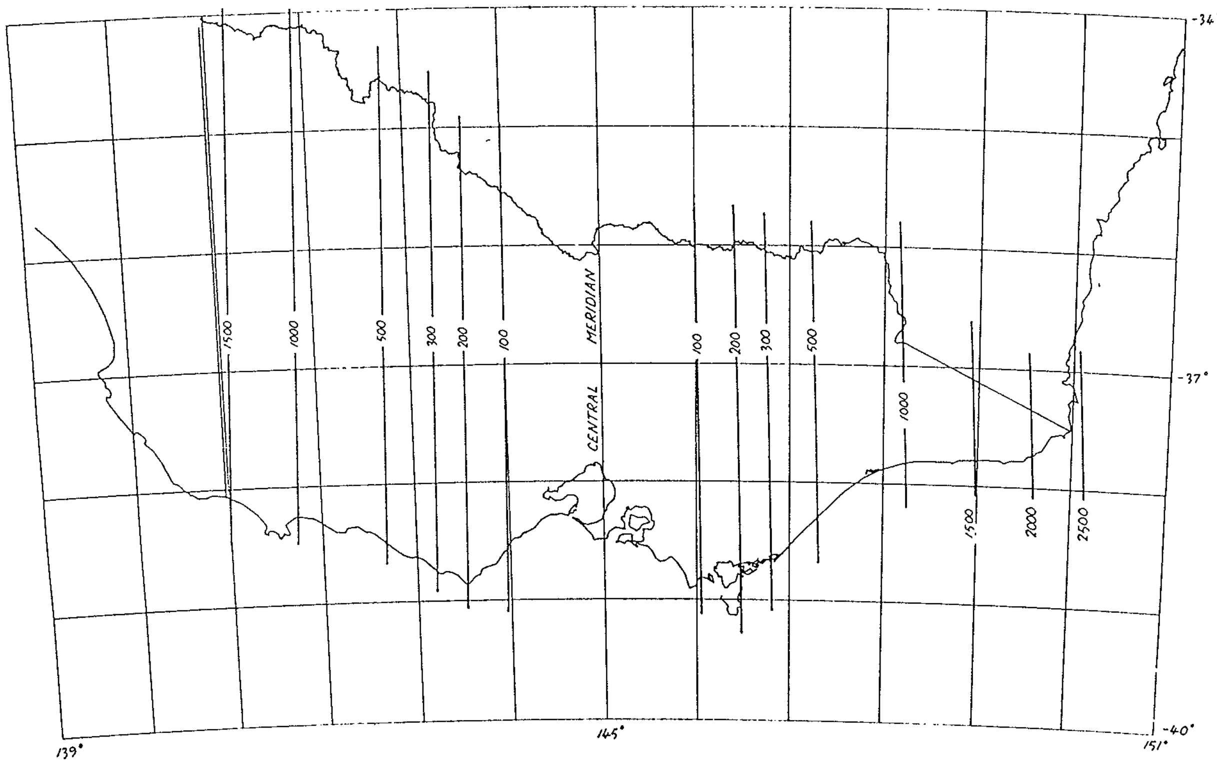

Figure 3. Transverse Mercator projection

BIBLIOGRAPHY

Airy, Sir G.B., 1861. "Explanation of a Projection by Balance of Errors ...." *Philosophical Magazine,* Vol. XXII, No. 149, pp. 409-421.

Canters, F. 1989., "New Projections for World Maps / A Quantitative Perceptive Approach." *Cartographica,* Vol. 26, No. 2, pp. 53-71.

———, 1991. "Map Projections in a GIS Environment." Proceedings of the 15th Conference of the International Cartographic Association, Bournemouth, UK, Sept. 1991, Vol. 2, pp. 595-604.

Dyer, J.A. and Snyder, J.P., 1989. "Minimum-Error Equal-Area Map Projections." *The American Cartographer,* Vol. 16, No. 1, pp. 39-43.

James, Sir H. and Clarke, A.R., 1862. "On Projections for Maps relating to a very large extent of the Earth's Surface." *Philosophical Magazine,* Vol. XXIII, pp. 306-312.

Lauf, G.B., 1983. *Geodesy and Map Projections.* TAFE Publications, Collingwood, Australia, 127 pages.

Nelder, J.A. and Mead, R., 1965. "A Simplex Method for Function Minimization." *Computer Journal,* Vol. 7, pp. 308-313.

Peters, A.B., 1984. "Distance-related Maps." *The American Cartographer,* Vol. 11, No. 2, pp. 119-131.

Press, W.H., Teukolsky, S.A., Vetterling, W.T. and Flannery, B.P., 1992. *Numerical Recipes in C - The Art of Scientific Computing.* 2nd Ed., Cambridge University Press, Cambridge, UK, 994 pages.

Reilly, W.I., 1973. "A Conformal Mapping Projection with Minimum Scale Error." *Survey Review,* Vol. 22, No. 168, pp. 57-71.

Snyder, J.P., 1984. "A Low-Error Conformal Map Projection for the 50 States." *The American Cartographer,* Vol. 11, No. 1, pp. 27-39.

———, 1985. *Computer-Assisted Map Projection Research.* U.S. Geological Survey Bulletin 1629, United States Government Printing Office, Washington, 157 pages.

———, 1987. *Map Projection - A Working Manual.* U.S. Geological Survey Professional Paper 1395, United States Government Printing Office, Washington, 383 pages.

Young, A.E., 1920. *Some Investigations in the Theory of map Projections.* Royal Geographical Society, London, 76 pages.